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# A relativistic extension of event-enhanced quantum theory

**Andreas Ruschhaupt**

Faculty of Physics, University of Bielefeld, Universitätsstrasse 25, D-33615 Bielefeld, Germany

E-mail: rushha@physik.uni-bielefeld.de

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**Abstract**

In this paper, we develop a formalism describing in a relativistic way a system which consists of a classical and a quantum part which are coupled. The formalism models one particle with spin $\frac{1}{2}$ and it is a possible relativistic extension of the event-enhanced quantum theory. We postulate a covariant algorithm which plays the role of the standard reduction postulate in non-relativistic quantum mechanics. Furthermore, we present an algorithm to simulate detections of the particle.

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## 1. Introduction

Seeking to bridge the conceptual frameworks of classical and quantum theory, Blanchard and Jadczyk [1–3] have proposed an extension of standard (non-relativistic) quantum mechanics called event-enhanced quantum theory (EEQT). Its main idea is to view the total system as consisting of a classical and a quantum part which are coupled. The pure states of the quantum part are wavefunctions which are not directly observable, whereas the pure states of the classical part can be observed without disturbing them. Changes of the classical pure states are called events. Events are discrete and irreversible. A review on applications of EEQT is, for example, [4].

Trying to define states and a reduction postulate in a relativistic theory can lead to paradoxes and logical difficulties (for example, see Aharonov and Albert [5]).

One possible solution is the assumption that there exists a preferred reference frame. For example, Caban and Rembieliński [6] used a non-standard synchronization scheme with built-in preferred reference frame and proposed a Poincaré-covariant relativistic quantum mechanics.

Furthermore, one possibility to avoid some of the above difficulties is to consider the wavefunction for a relativistic particle not as a function on the spacetime continuum but as a

  

function on the set of flat, space-like hypersurfaces in Minkowski space (for example, see the papers by Breuer and Petruccione [7–11]).

Another possibility is the introduction of a supplementary, intrinsic time, the proper time $\tau$. The proper time $\tau$ is independent of the reference frame. It plays the role of (absolute) time in non-relativistic quantum mechanics. The wavefunction at a given proper time is a function on the spacetime continuum. The idea of a proper time was first used physically by Horwitz and Piron [12] and later in a lot of other approaches (a review with more references is, for example, written by Fanchi [13]).

Blanchard and Jadczyk have also introduced a relativistic version of EEQT [14] using the idea of a proper time and an indefinite scalar product.

The main aim of this paper is to present an alternative relativistic version of EEQT which uses a positive-definite scalar product. The theory will describe a single spin-$\frac{1}{2}$ particle with mass $m$ in a relativistic way and should be useful in situations in which one can neglect pair-creation and pair-annihilation. As in the relativistic extension introduced by Blanchard and Jadczyk [14], we postulate an additional parameter, called proper time $\tau$. The total system consists of a classical and a quantum part. Therefore, at a given proper time $\tau$, the (pure) state of the total system is a pair $(\omega_\tau, \Psi_\tau)$. $\omega_\tau$ is the state of the classical part and $\Psi_\tau$ is the state of the quantum part.

We assume that a (pure) state $\omega_\tau$ of the classical part is a number: $\omega_\tau \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Again, a change of the classical (pure) state is called an 'event'.

The (pure) states of the quantum part shall be (heuristically speaking) solutions $\Psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ of the Dirac equation $\left(\mathrm{i}\gamma^\mu \partial_\mu - \frac{e}{c\hbar}\gamma^\mu A_\mu - \frac{mc}{\hbar}\right)\Psi(x) = 0$. The quantum state is always a function on the spacetime continuum, even if ideal or continuous measurements are performed. An interesting property of a quantum state is that it is uniquely given by its values on a space-like hyperplane. Moreover, it will be possible to introduce a positive-definite scalar product between two quantum states. In section 2, we present the definition of (pure) states of the quantum part and their properties in a more mathematical way.

We define in section 3 how the system state changes if we change the reference frame or 'charge conjugate' the system.

In section 4, we postulate a covariant algorithm for simulating ideal measurements of infinitesimally small duration. It plays the role of the standard reduction postulate in non-relativistic quantum mechanics.

An algorithm for simulating detections of the particle is presented and examined in section 5.

In the last section, we summarize the properties of our formalism.

In a future paper, we will examine applications of our algorithm for simulating detections. A first application can be found in [15].

## 2. Pure states of the quantum part

We want to define a (pure) state of the quantum part of the total system. It describes the state of a single particle with spin $\frac{1}{2}$ and mass $m$.

Let $\mathcal{P} = \{(y, \vec{\alpha}, \vec{\varphi}) : y \in \mathbb{R}^4, \vec{\alpha} \in \mathbb{R}^3, |\vec{\alpha}| < 1, \vec{\varphi} \in \mathbb{R}^3, |\vec{\varphi}| < \pi\}$ and we define with $\lambda \equiv ((y^0, \vec{y}), \vec{\alpha}, \vec{\varphi}) \in \mathcal{P}$:

$$\sigma_\lambda(\vec{u}) = (y^0 + \vec{\alpha} \cdot \hat{R}_{\vec{\varphi}}\vec{u}, \vec{y} + \hat{R}_{\vec{\varphi}}\vec{u}) \qquad \forall \vec{u} \in \mathbb{R}^3$$

$$\langle f|g\rangle_\lambda = \int \mathrm{d}\vec{u}\, f^+(\vec{u})(1 - \gamma^0\vec{\gamma}\vec{\alpha})g(\vec{u}) \qquad \forall f, g \in L_2(\mathbb{R}^3)^4$$

$$\|f\|_\lambda = \sqrt{\langle f|f\rangle_\lambda} \qquad \forall f \in L_2(\mathbb{R}^3)^4$$

with $\bar{\mathbb{C}} = \mathbb{C} \cup \{\pm\infty + \mathrm{i}\mathbb{R}\} \cup \{\mathbb{R} \pm \mathrm{i}\infty\} \cup \{\pm\infty \pm \mathrm{i}\infty\}$ and $L_2(\mathbb{R}^3)^4 = \{f : \mathbb{R}^3 \to \bar{\mathbb{C}}^4 : \int \mathrm{d}x |f(x)|^2 < \infty\}$. $\gamma^\mu = (\gamma^0, \vec{\gamma})$ are the Dirac matrices and $\hat{R}_{\vec{\varphi}} \in SO(3)$ is the rotation of the angle $|\vec{\varphi}|$ around the vector $\vec{\varphi}/|\vec{\varphi}|$ (the sense of rotation is determined by the right-hand rule). Note that this scalar product was also used by Breuer and Petruccione [7–11].

We continue with the following definition.

**Definition.** $\Psi \in \hat{\mathcal{H}}$ *if and only if the following conditions are satisfied*

*(i)* $\Psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4$ $\Psi$ *continuous differentiable* (1)

*(ii)* $\left(\mathrm{i}\gamma^\mu \partial_\mu - \frac{e}{c\hbar}\gamma^\mu A_\mu - \frac{mc}{\hbar}\right)\Psi(x) = 0$ (2)

*(iii)* $\|\Psi \circ \sigma_\lambda\|_\lambda < \infty$ *for all* $\lambda \in \mathcal{P}$ (3)

*(iv)* $\lim_{|\vec{u}|\to\infty} |\vec{u}|^3 |\Psi \circ \sigma_\lambda(\vec{u})|^2 = 0$ *for all* $\lambda \in \mathcal{P}$ (4)

$A_\mu : \mathbb{R}^4 \to \mathbb{R}^4$ *is the external electromagnetic potential.*

$\hat{\mathcal{H}}$ is a vector space. Now we want to define a scalar product for all $\Psi \in \hat{\mathcal{H}}$. The following theorem is very important for achieving this task.

**Theorem 1.** *Let* $\Psi_A, \Psi_B \in \hat{\mathcal{H}}$, *let* $j^\mu_{AB} := \Psi^+_A \gamma^0 \gamma^\mu \Psi_B$, *the quantity*

$$\langle \Psi_A \circ \sigma_\lambda | \Psi_B \circ \sigma_\lambda \rangle_\lambda \equiv \int_{\sigma_\lambda} j^\mu_{AB}\, \mathrm{d}f_\mu$$

*exists for all* $\lambda = ((y^0, \vec{y}), \vec{\alpha}, \vec{\varphi}) \in \mathcal{P}$ *and is independent of* $\lambda$. $\mathrm{d}f_\mu \equiv (1, -\vec{\alpha})\,\mathrm{d}\vec{u}$ *denotes the differential 'surface element' of* $\sigma_\lambda$.

**Proof.**

(i) *Existence.* This follows from the fact that $\Psi_A \circ \sigma_\lambda$, $\Psi_B \circ \sigma_\lambda \in L_2(\mathbb{R}^3)^4$ (see (3)).

(ii) *Independence.* We get $\partial_\mu j^\mu_{AB} = 0$ by a simple calculation. The integral is clearly independent of $\vec{\varphi}$ and $\vec{y}$. Therefore, we can assume $\vec{\varphi} = 0$ and $\vec{y} = 0$. Let $\sigma_1 = \sigma_{((y^0_1,\vec{0}),\vec{\alpha}_1,0)}$ and $\sigma_2 = \sigma_{((y^0_2,\vec{0}),\vec{\alpha}_2,0)}$ be two hyperplanes. Let $\hat{x}(\varphi, \Theta) = (\cos\varphi \sin\Theta, \sin\varphi \sin\Theta, \cos\Theta)$.

(a) *Case* $\vec{\alpha}_1 = \vec{\alpha}_2 =: \vec{\alpha}$. Let

$$F_1(R) = \{\sigma_1(\vec{u}) : |\vec{u}| \leqslant R\}$$
$$F_2(R) = \{\sigma_2(\vec{u}) : |\vec{u}| \leqslant R\}$$
$$s_R(\nu, \varphi, \Theta) = \left(y^0_1 + \nu\left(y^0_2 - y^0_1\right) + R \cdot \vec{\alpha}\hat{x}(\varphi, \Theta), R \cdot \hat{x}(\varphi, \Theta)\right)$$
$$S(R) = \{s_R(\nu, \varphi, \Theta) : 0 \leqslant \nu \leqslant 1, 0 \leqslant \varphi < 2\pi, 0 \leqslant \Theta < \pi\}.$$

Let $V(R)$ be the volume bounded by $F_1(R)$, $F_2(R)$ and $S(R)$. The differential 'surface element' of $S(R)$ is $\mathrm{d}S_\mu = R^2 W_\mu(\nu, \varphi, \alpha)\,\mathrm{d}\nu\,\mathrm{d}\varphi\,\mathrm{d}\Theta$. The function $W_\mu$ need not be explicitly calculated, because it is enough to know that $W_\mu$ does not depend on $R$. We get by the Gauss theorem $\left(\text{with } j^\mu_{AB}(R, \nu, \varphi, \Theta) \equiv j^\mu_{AB} \circ s_R(\nu, \varphi, \Theta)\right)$

$$
\begin{aligned}
-\int_{\sigma_1} j^\mu_{AB}\,\mathrm{d}f_\mu + \int_{\sigma_2} j^\mu_{AB}\,\mathrm{d}f_\mu &= -\lim_{R\to\infty}\int_{F_1(R)} j^\mu_{AB}\,\mathrm{d}f_\mu + \lim_{R\to\infty}\int_{F_2(R)} j^\mu_{AB}\,\mathrm{d}f_\mu \\
&= \lim_{R\to\infty}\int_{V(R)} \partial_\mu j^\mu_{AB}\,\mathrm{d}^4x - \lim_{R\to\infty}\int_{S(R)} j^\mu_{AB}\,\mathrm{d}S_\mu \\
&= -\lim_{R\to\infty}\int_{S(R)} j^\mu_{AB}\,\mathrm{d}S_\mu \\
&= -\lim_{R\to\infty}\int \mathrm{d}\nu \int \mathrm{d}\varphi \int \mathrm{d}\Theta R^2 j^\mu_{AB}(R,\nu,\varphi,\Theta) W_\mu(\nu,\varphi,\alpha) \\
&= -\int \mathrm{d}\nu \int \mathrm{d}\varphi \int \mathrm{d}\Theta \lim_{R\to\infty}\left(R^2 j^\mu_{AB}(R,\nu,\varphi,\Theta)\right) W_\mu(\nu,\varphi,\alpha) = 0
\end{aligned}
$$

because

$$
\begin{aligned}
R^2\left|j^\mu_{AB}(R,\nu,\varphi,\Theta)\right| &= R^2\left|\Psi^+_A\gamma^0\gamma^\mu\Psi_B\circ s_R(\nu,\varphi,\Theta)\right| \\
&\leqslant \frac{\text{const}}{2}(R^2|\Psi_A\circ s_R(\nu,\varphi,\Theta)|^2 + R^2|\Psi_B\circ s_R(\nu,\varphi,\Theta)|^2) \overset{R\to\infty}{\longrightarrow} 0
\end{aligned}
$$

uniformly in $\varphi$, $\Theta$ (see (4)) and $\nu$ (because $j^\mu_{AB}$ is continuous).

(b) *Case* $\vec{\alpha}_1 \neq \vec{\alpha}_2$. Because of case (a), we can assume $y^0_1 = y^0_2 = 0$. Let $\vec{\alpha}(\nu)$ be chosen in such a way that $\vec{\alpha}(\nu)$ is continuous, $\vec{\alpha}(0) = \vec{\alpha}_1, \vec{\alpha}(1) = \vec{\alpha}_2$ and $|\vec{\alpha}(\nu)| < 1\ \forall\nu \in [0,1]$. Let

$$
\begin{aligned}
F_1(R) &= \{\sigma_1(\vec{u}) : |\vec{u}| \leqslant R\} \\
F_2(R) &= \{\sigma_2(\vec{u}) : |\vec{u}| \leqslant R\} \\
s_R(\nu,\varphi,\Theta) &= (R\cdot\vec{\alpha}(\nu)\hat{x}(\varphi,\Theta), R\cdot\hat{x}(\varphi,\Theta)) \\
S(R) &= \{s_R(\nu,\varphi,\Theta) : 0 \leqslant \nu \leqslant 1, 0 \leqslant \varphi < 2\pi, 0 \leqslant \Theta < \pi\}.
\end{aligned}
$$

Again, $V(R)$ should be the volume bounded by $F_1(R)$, $F_2(R)$ and $S(R)$. The differential 'surface element' of $S(R)$ is $\mathrm{d}S_\mu = R^3\tilde{W}_\mu(\nu,\varphi,\alpha)\,\mathrm{d}\nu\,\mathrm{d}\varphi\,\mathrm{d}\Theta$ (note the factor $R^3$ instead of $R^2$ in case (a)!). Analogous to case (a), it follows

$$
-\int_{\sigma_1} j^\mu_{AB}\,\mathrm{d}f_\mu + \int_{\sigma_2} j^\mu_{AB}\,\mathrm{d}f_\mu = 0
$$

because $\left|R^3 j^\mu_{AB}(R,\nu,\varphi,\Theta)\right| \overset{R\to\infty}{\longrightarrow} 0$ uniformly in $\nu$, $\varphi$, $\Theta$. □

Now we are able to introduce a scalar product between elements of $\hat{\mathcal{H}}$.

**Definition.** *We introduce a scalar product between* $\Psi_A, \Psi_B \in \hat{\mathcal{H}}$*:*

$$
\langle\Psi_A|\Psi_B\rangle_{\hat{\mathcal{H}}} := \langle\Psi_A\circ\sigma_\lambda|\Psi_B\circ\sigma_\lambda\rangle_\lambda \qquad \|\Psi_A\|_{\hat{\mathcal{H}}} := \sqrt{\langle\Psi_A|\Psi_A\rangle_{\hat{\mathcal{H}}}}
$$

*with* $\lambda \in \mathcal{P}$ *arbitrary.*

$\langle\cdot|\cdot\rangle_{\hat{\mathcal{H}}}$ is a sesquilinear form. It is clear that $\langle\Psi|\Psi\rangle_{\hat{\mathcal{H}}} \geqslant 0\ \forall\Psi \in \hat{\mathcal{H}}$ because the eigenvalues of $(1-\gamma^0\vec{\gamma}\vec{\alpha})$ are $1+|\vec{\alpha}| > 0$ and $1-|\vec{\alpha}| > 0$.

The independence of the scalar product from the parameters $\lambda \equiv (y,\vec{\alpha},\vec{\varphi})$ 'expresses' the independence of the reference frame. Note that the number of 'free parameters' is ten and equals the number of parameters of a Poincaré transformation.

An element $\Psi \in \hat{\mathcal{H}}$ is uniquely given by its values on a hyperplane $\sigma_\lambda$. This fact results indeed from the following theorem.

**Theorem 2.** *Let $\mu = (y, \vec{\alpha}, \vec{\varphi}) \in \mathcal{P}$ arbitrary, let $\Psi_1, \Psi_2 \in \hat{\mathcal{H}}$ with $\Psi_1 \circ \sigma_\mu = \Psi_2 \circ \sigma_\mu$ then it follows $\Psi_1 = \Psi_2$.*

**Proof.** Let $\Psi := \Psi_1 - \Psi_2$, we get $\Psi \circ \sigma_\mu(\vec{u}) = 0 \; \forall \vec{u}$ and therefore $\|\Psi \circ \sigma_\mu\|_\mu = 0$. We assume $\Psi_1 \neq \Psi_2$, so there exists $z = (z^0, \vec{z}) \in \mathbb{R}^4$ with $\Psi(z) = \Psi_1(z) - \Psi_2(z) \neq 0$. Because $\Psi$ is continuous there must be a neighbourhood of $z$ with $\Psi(x) \neq 0$. So there exists $\epsilon > 0$ with $\Psi \circ \sigma_{(z,\vec{\alpha},\vec{\varphi})}(\vec{u}) \neq 0$ for all $\vec{u}$ with $|\vec{u}| < \epsilon$ (because $z = \sigma_{(z,\vec{\alpha},\vec{\varphi})}(0)$). It follows that $\|\Psi \circ \sigma_{(z,\vec{\alpha},\vec{\varphi})}\|_{\sigma_{(z,\vec{\alpha},\vec{\varphi})}} > 0$. But we get

$$0 = \|\Psi \circ \sigma_{(y,\vec{\alpha},\vec{\varphi})}\|_{\sigma_{(y,\vec{\alpha},\vec{\varphi})}} \overset{\text{theorem 1}}{=} \|\Psi \circ \sigma_{(z,\vec{\alpha},\vec{\varphi})}\|_{\sigma_{(z,\vec{\alpha},\vec{\varphi})}} \neq 0.$$

The assumption that $\Psi_1 \neq \Psi_2$ is wrong and it implies that $\Psi_1 = \Psi_2$. □

**Theorem 3.** *$(\hat{\mathcal{H}}, \langle\cdot|\cdot\rangle_{\hat{\mathcal{H}}})$ is a pre-Hilbert space.*

**Proof.** It is only left to prove that $\langle\Psi|\Psi\rangle_{\hat{\mathcal{H}}} = 0$ provides $\Psi = 0$. We assume

$$0 = \langle\Psi(0)|\Psi(0)\rangle_{\hat{\mathcal{H}}} = \int \mathrm{d}\vec{u}\, |\Psi(0, \vec{u})|^2 = \langle\Psi|\Psi\rangle_{L_2(\mathbb{R}^3)^4}.$$

It results that $\Psi(0, \vec{u}) = 0 \; \forall \vec{u}$, because $\langle\cdot|\cdot\rangle_{L_2(\mathbb{R}^3)^4}$ is a scalar product. As $0(0, \vec{u}) = 0 \; \forall \vec{u}$ and $\Psi(0, \vec{u}) = 0 \; \forall \vec{u}$, we get by theorem 2 that $\Psi = 0$. □

We demand that the quantum states are elements of a Hilbert space. So we must complete the pre-Hilbert space $(\hat{\mathcal{H}}, \langle\cdot|\cdot\rangle_{\hat{\mathcal{H}}})$.

**Definition.** *Let*

$$\mathcal{H} = \{F : \mathbb{R}^4 \to \bar{\mathbb{C}}^4 | F \circ \sigma_\lambda \in L_2(\mathbb{R}^3)^4 \; \forall \lambda \in \mathcal{P} \text{ and } \exists \text{ sequence } \{\Psi_m\}_{m\in\mathbb{N}}, \Psi_m \in \hat{\mathcal{H}} : \\ \forall \epsilon > 0 \; \exists N_\epsilon : \|(F - \Psi_m) \circ \sigma_\lambda\|_\lambda < \epsilon \quad \forall m > N_\epsilon \; \forall \lambda \in \mathcal{P}\}. \tag{5}$$

*Let $F \in \mathcal{H}$, we define $F = 0 \Leftrightarrow \|F \circ \sigma_\lambda\|_\lambda = 0 \; \forall \lambda \in \mathcal{P}$.*

*A scalar product $\langle\cdot|\cdot\rangle_{\mathcal{H}} : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ is defined by*

$$\langle F_1|F_2\rangle_{\mathcal{H}} := \langle F_1 \circ \sigma_\lambda | F_2 \circ \sigma_\lambda\rangle_\lambda \qquad \forall F_1, F_2 \in \mathcal{H}$$

*with $\lambda \in \mathcal{P}$ arbitrary.*

The following theorem proves that $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ is really a Hilbert space and a completion of $(\hat{\mathcal{H}}, \langle\cdot|\cdot\rangle_{\hat{\mathcal{H}}})$.

**Theorem 4.** *The above scalar product is well defined (independent of the parameter $\lambda$). $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ is a Hilbert space and $\hat{\mathcal{H}}$ is a dense subspace of it.*

**Proof.** (i) We first prove that $\mathcal{H}$ is a vector space. The only thing which is (perhaps) not trivial is the existence of a sequence in the above sense. Let $F_1, F_2 \in \mathcal{H}, a, b \in \mathbb{C}$, then there exist sequences $\Psi_{1,m}, \Psi_{2,m}$ in the above sense. Now we get

$$\begin{aligned}\|((aF_1 + bF_2) - (a\Psi_{1,m} + b\Psi_{2,m})) \circ \sigma_\lambda\|_\lambda &= \|a(F_1 - \Psi_{1,m}) \circ \sigma_\lambda + b(F_2 - \Psi_{2,m}) \circ \sigma_\lambda\|_\lambda \\ &\leqslant |a|\|(F_1 - \Psi_{1,m}) \circ \sigma_\lambda\|_\lambda + |b|\|(F_2 - \Psi_{2,m}) \circ \sigma_\lambda\|_\lambda \\ &\overset{m\to\infty}{\longrightarrow} 0\end{aligned}$$

uniformly for all $\lambda \in \mathcal{P}$.

(ii) We now prove that $\langle F_1 \circ \sigma_\lambda | F_2 \circ \sigma_\lambda \rangle_\lambda$ is independent of $\lambda \in \mathcal{P}$ for all $F_1, F_2 \in \mathcal{H}$. Let $F_1, F_2 \in \mathcal{H}$, then there exist sequences $\Psi_{1,m}, \Psi_{2,m}$ in the above sense. Now we get with $\lambda, \bar{\lambda} \in \mathcal{P}$

$$\begin{aligned}
\langle F_1 \circ \sigma_\lambda | F_2 \circ \sigma_\lambda \rangle_\lambda &= \langle \lim -\lambda_{m\to\infty}(\Psi_{1,m} \circ \sigma_\lambda) | \lim -\lambda_{m\to\infty}(\Psi_{2,m} \circ \sigma_\lambda) \rangle_\lambda \\
&= \lim_{m\to\infty} \langle \Psi_{1,m} \circ \sigma_\lambda | \Psi_{2,m} \circ \sigma_\lambda \rangle_\lambda \\
&\overset{\text{theorem 1}}{=} \lim_{m\to\infty} \langle \Psi_{1,m} \circ \sigma_{\bar{\lambda}} | \Psi_{2,m} \circ \sigma_{\bar{\lambda}} \rangle_{\bar{\lambda}} \\
&= \langle \lim -\bar{\lambda}_{m\to\infty}(\Psi_{1,m} \circ \sigma_{\bar{\lambda}}) | \lim -\bar{\lambda}_{m\to\infty}(\Psi_{2,m} \circ \sigma_{\bar{\lambda}}) \rangle_{\bar{\lambda}} \\
&= \langle F_1 \circ \sigma_{\bar{\lambda}} | F_2 \circ \sigma_{\bar{\lambda}} \rangle_{\bar{\lambda}}.
\end{aligned}$$

$\lim -\lambda$ and $\lim -\bar{\lambda}$ mean the limits concerning the norms $\|\cdot\|_\lambda$ and $\|\cdot\|_{\bar{\lambda}}$ in the Hilbert spaces $(L_2(\mathbb{R}^3)^4, \langle\cdot|\cdot\rangle_\lambda)$ and $(L_2(\mathbb{R}^3)^4, \langle\cdot|\cdot\rangle_{\bar{\lambda}})$ . It is now clear that $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is a sesquilinear form with $\langle F|F\rangle_{\mathcal{H}} \geqslant 0 \; \forall F \in \mathcal{H}$.

(iii) Let $F \in \mathcal{H}$ with $\langle F|F\rangle_{\mathcal{H}} = 0$, then it follows

$$\begin{aligned}
& \langle F|F\rangle_{\mathcal{H}} = 0 \\
\Rightarrow \quad & \langle F \circ \sigma_\lambda | F \circ \sigma_\lambda \rangle_\lambda = 0 \qquad \forall \lambda \in \mathcal{P} \\
\Rightarrow \quad & \|F \circ \sigma_\lambda\|_\lambda = 0 \qquad \forall \lambda \in \mathcal{P} \\
(\Leftrightarrow \quad & (F \circ \sigma_\lambda)(\vec{u}) = 0 \quad \text{for almost all } \vec{u} \in \mathbb{R}^3, \; \forall \lambda \in \mathcal{P}) \\
\overset{\text{def.}}{\Leftrightarrow} \quad & F = 0.
\end{aligned}$$

So $\langle\cdot|\cdot\rangle_{\mathcal{H}}$ is a scalar product.

(iv) We now prove that $\mathcal{H}$ is complete. Let $\{F_m\}$ be a Cauchy sequence in $\mathcal{H}$, so $\|(F_m - F_n) \circ \sigma_\lambda\|_\lambda \overset{m,n\to\infty}{\longrightarrow} 0$ uniformly $\forall \lambda \in \mathcal{P}$. So there exists a sub-sequence $\{F_{n_k}\}$ with

$$\left\| \left(F_{n_{k+1}} - F_{n_k}\right) \circ \sigma_\lambda \right\|_\lambda \leqslant 2^{-k} \qquad \forall k \, \forall \lambda \in \mathcal{P}.$$

By using Lebesgue's dominated convergence theorem with the sequence $f_{\lambda,k} := F_{n_k} \circ \sigma_\lambda$, we can show that there exists $f_\lambda \in L_2(\mathbb{R}^3)^4$ with

$$\left\| F_{n_k} \circ \sigma_\lambda - f_\lambda \right\|_\lambda \overset{k\to\infty}{\longrightarrow} 0 \qquad \forall \lambda \in \mathcal{P}$$

and $\lim_{k\to\infty} F_{n_k} \circ \sigma_\lambda(\vec{u}) = f_\lambda(\vec{u})$ almost everywhere and $\forall \lambda \in \mathcal{P}$. Note that the sub-sequence $\{n_k\}_{k\in\mathbb{N}}$ is independent of $\lambda$! We also get $\|F_n \circ \sigma_\lambda - f_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0 \; \forall \lambda \in \mathcal{P}$. Because $\|(F_m - F_n) \circ \sigma_\lambda\|_\lambda \overset{m,n\to\infty}{\longrightarrow} 0$ uniformly $\forall \lambda \in \mathcal{P}$, we get by taking $\lim -\lambda_{m\to\infty}$ that $\|F_n \circ \sigma_\lambda - f_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0$ uniformly $\forall \lambda \in \mathcal{P}$. Now we set

$$F(x) = \begin{cases} f_\lambda(\vec{u}) & \text{if} \quad \sigma_\lambda(\vec{u}) = x \quad \text{and} \quad \lim_{k\to\infty} F_{n_k} \circ \sigma_\lambda(\vec{u}) = f_\lambda(\vec{u}) \\ 0 & \text{otherwise.} \end{cases}$$

This function is well defined, because the sub-sequence $F_{n_k}$ is independent of $\lambda$! It is also trivial that $F \circ \sigma_\lambda = f_\lambda$ almost everywhere.

We now prove that $F \in \mathcal{H}$. The only thing left to prove is the existence of a sequence $\Psi_m \in \hat{\mathcal{H}}$. Because $F_m \in \mathcal{H}$, there exist sequences $\{\Phi_{m,v}\}$ with $\|(F_m - \Phi_{m,v}) \circ \sigma_\lambda\|_\lambda \overset{v\to\infty}{\longrightarrow} 0$ uniformly $\forall \lambda \in \mathcal{P}$. So there exists $\Psi_m \in \hat{\mathcal{H}}$ with $\|(F_m - \Psi_m) \circ \sigma_\lambda\|_\lambda < \frac{1}{m} \; \forall \lambda \in \mathcal{P}$. Now we get

$$\begin{aligned}
\|(F - \Psi_m) \circ \sigma_\lambda\|_\lambda &\leqslant \|(F - F_m) \circ \sigma_\lambda\|_\lambda + \|(F_m - \Psi_m) \circ \sigma_\lambda\|_\lambda \\
&\leqslant \underbrace{\|f_\lambda - F_m \circ \sigma_\lambda\|_\lambda}_{\overset{m\to\infty}{\longrightarrow} 0 \text{ uniformly } \forall \lambda \in \mathcal{P}} + \frac{1}{m} \overset{m\to\infty}{\longrightarrow} 0
\end{aligned}$$

uniformly(!) for all $\lambda \in \mathcal{P}$. So it results that $F \in \mathcal{H}$.

The last step to prove is: $F_m \overset{m\to\infty}{\longrightarrow} 0$ concerning the norm in $\mathcal{H}$. We get for all $\lambda \in \mathcal{P}$

$$\|(F - F_m) \circ \sigma_\lambda\|_\lambda = \|f_\lambda - F_m \circ \sigma_\lambda\|_\lambda \overset{m\to\infty}{\longrightarrow} 0.$$

(v) It is trivial that $\hat{\mathcal{H}} \subset \mathcal{H}$ and that $\hat{\mathcal{H}}$ is dense in $\mathcal{H}$. □

We are now in a position to postulate that the (pure) states of the quantum part of the total system are the elements of the Hilbert space $(\mathcal{H}, \langle\cdot|\cdot\rangle_\mathcal{H})$. An important fact is that the elements are still functions on the spacetime continuum.

Let $\lambda \in \mathcal{P}$ and we define the function $U_\lambda : \mathcal{H} \to \mathcal{R}_\lambda$ by

$$U_\lambda : \mathcal{H} \ni F(x) \quad \longmapsto \quad F \circ \sigma_\lambda(\vec{u}) \in \mathcal{R}_\lambda \tag{6}$$

with $\mathcal{R}_\lambda \subset L_2(\mathbb{R}^3)^4$ denoting the range of $U_\lambda$. A quantum state is uniquely given by its values on a hyperplane $\sigma_\lambda$. This means that the function $U_\lambda$ is injective for all $\lambda \in \mathcal{P}$. The following theorem proves this property.

**Theorem 5.** *Let $F_1, F_2 \in \mathcal{H}$ such that there exists $\mu \in \mathcal{P}$ with $F_1 \circ \sigma_\mu = F_2 \circ \sigma_\mu$ then it follows $F_1 = F_2$.*

**Proof.** Let $F = F_1 - F_2$, we get $F \circ \sigma_\mu = 0$. $F \in \mathcal{H}$, so there exists a sequence $\Psi_n \in \hat{\mathcal{H}}$ with $\|(F - \Psi_n) \circ \sigma_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0$ uniformly for all $\lambda \in \mathcal{P}$. We get

$$0 \overset{n\to\infty}{\longleftarrow} \|(F - \Psi_n) \circ \sigma_\mu\|_\mu = \|\Psi_n \circ \sigma_\mu\|_\mu \overset{\text{theorem 1}}{=} \|\Psi_n \circ \sigma_\lambda\|_\lambda \qquad \forall\lambda \in \mathcal{P}.$$

Because

$$\|F \circ \sigma_\lambda\|_\lambda \leqslant \|(F - \Psi_n) \circ \sigma_\lambda\|_\lambda + \|\Psi_n \circ \sigma_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0 \qquad \forall\lambda \in \mathcal{P}$$

we get $\|F \circ \sigma_\lambda\|_\lambda = 0 \,\forall\lambda \in \mathcal{P} \Leftrightarrow F = 0$.
It follows that $0 = F = F_1 - F_2 \Rightarrow F_1 = F_2$. □

The function $U_\lambda$ is invertible, let $U_\lambda^{-1} : \mathcal{R}_\lambda \to \mathcal{H}$ be the inverse function. The following theorem proves some properties of $U_\lambda$ and $U_\lambda^{-1}$ respectively.

**Theorem 6.** *Let $\lambda \in \mathcal{P}$ and the functions $U_\lambda : \mathcal{H} \to \mathcal{R}_\lambda$ and $U_\lambda^{-1} : \mathcal{R}_\lambda \to \mathcal{H}$ are defined as above.*

*(i) Let $F \in \mathcal{H}$ and $f \in \mathcal{R}_\lambda$, then $\left\langle U_\lambda^{-1} f \middle| F \right\rangle_\mathcal{H} = \langle f | U_\lambda F\rangle_\lambda$ and especially $\|U_\lambda F\|_\lambda = \|F\|_\mathcal{H}$ and $\left\|U_\lambda^{-1} f\right\|_\mathcal{H} = \|f\|_\lambda$.*

*(ii) We use the Hilbert space $(L_2(\mathbb{R}^3)^4, \langle\cdot|\cdot\rangle_\lambda)$. Then the set $\mathcal{R}_\lambda \subset L_2(\mathbb{R}^3)^4$ is closed.*

**Proof.**

$$\text{(i)}\ \left\langle U_\lambda^{-1} f \middle| F \right\rangle_\mathcal{H} = \Big\langle \underbrace{\left(U_\lambda^{-1} f\right) \circ \sigma_\lambda}_{f} \Big| \underbrace{F \circ \sigma_\lambda}_{U_\lambda F} \Big\rangle_\lambda = \langle f | U_\lambda F\rangle_\lambda$$

(ii) Let $f_n \in \mathcal{R}_\lambda$ for all $n \in \mathbb{N}$ and $\lim -\lambda_{n\to\infty} f_n = f \in L_2(\mathbb{R}^3)^4$. We want to prove that $f \in \mathcal{R}_\lambda$. We set $F_n := U_\lambda^{-1} f_n$. $\{F_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Because $\mathcal{H}$ is complete, there exists $F \in \mathcal{H}$ with $\lim_{n\to\infty} F_n = F$. Moreover we get

$$f = \lim -\lambda_{n\to\infty} f_n = \lim -\lambda_{n\to\infty} U_\lambda F_n = U_\lambda \lim_{n\to\infty} F_n = U_\lambda F$$

(because $U_\lambda$ is bounded/continuous). So $\mathcal{R}_\lambda$ is closed. □

It follows that $(\mathcal{R}_\lambda, \langle\cdot|\cdot\rangle_\lambda)$ is a Hilbert space and $U_\lambda : (\mathcal{H}, \langle\cdot|\cdot\rangle_\mathcal{H}) \to (\mathcal{R}_\lambda, \langle\cdot|\cdot\rangle_\lambda)$ is a unitary operator for all $\lambda \in \mathcal{P}$. Because $(L_2(\mathbb{R}^3)^4, \langle\cdot|\cdot\rangle_\lambda)$ is a separable Hilbert space,

$(\mathcal{R}_\lambda, \langle\cdot|\cdot\rangle_\lambda)$ is a separable 'sub'-Hilbert space. Therefore, $(\mathcal{H}, \langle\cdot|\cdot\rangle_{\mathcal{H}})$ must be a separable Hilbert space.

Before continuing, let us compare our approach with that of Breuer and Petruccione [7–11]. They introduced the Hilbert bundle $\mathcal{H}_{BP}(\lambda) \equiv (L_2(\mathbb{R}^3)^4, \langle\cdot|\cdot\rangle_\lambda)$ and a wavefunction $\psi_{BP}$ which depends on the hyperplane, $\psi_{BP}(\lambda) \in \mathcal{H}_{BP}(\lambda)$. Theorem 6 gives the result that $\mathcal{H}$ is isomorph to $\mathcal{H}_{BP}(\bar{\lambda})$ for a fixed (but arbitrary) $\bar{\lambda} \in \mathcal{P}$. Without measurements, the wavefunction $\psi_{BP}$ is more or less a function on the spacetime continuum, i.e. there exists a function $\Psi(x)$ with $\psi_{BP}(\lambda) = \Psi \circ \sigma_\lambda$. This function $\Psi$ is the quantum state in our formalism. But this connection is not true in the case of measurements.

## 3. Change of the reference frame and charge conjugation

Our aim now is to define how the quantum state changes, if we change the reference frame $K \to \tilde{K}$ with $\tilde{x} = \Lambda x + a$. The classical state does not change in this case.

We look only at Poincaré transformations $(\Lambda, a)$ which do not mirror the space and do not invert the direction of time, i.e., $\Lambda \in L_+^\uparrow$. Let $S(\Lambda)$ be a non-singular $4 \times 4$ matrix with $S^{-1}(\Lambda)\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu$, $S^{-1}(\Lambda) = S(\Lambda^{-1})$ and $S^{-1}(\Lambda) = \gamma^0 S^+(\Lambda)\gamma^0$.

Let us first present a lemma which will be needed in the proofs of the main theorems.

**Lemma.** *Let $f, g : \mathbb{R}^4 \to \bar{\mathbb{C}}^4$, $f \circ \sigma_\lambda, g \circ \sigma_\lambda \in L_2(\mathbb{R}^3)^4$ for all $\lambda \in \mathcal{P}$, $\Lambda \in L_+^\uparrow$, $a \in \mathbb{R}^4$, we set*

$$\tilde{f}(\tilde{x}) = S(\Lambda) f(\Lambda^{-1}(\tilde{x} - a)) \qquad \tilde{g}(\tilde{x}) = S(\Lambda) g(\Lambda^{-1}(\tilde{x} - a)).$$

*Let $\lambda \in \mathcal{P}$ arbitrary, then there exists $\mu(\lambda) \in \mathcal{P}$ with*

$$\langle \tilde{f} \circ \sigma_\lambda | \tilde{g} \circ \sigma_\lambda \rangle_\lambda = \langle f \circ \sigma_{\mu(\lambda)} | g \circ \sigma_{\mu(\lambda)} \rangle_{\mu(\lambda)}.$$

**Proof.** Each arbitrary Lorentz transformation $\Lambda \in L_+^\uparrow$ can be expressed as a product of pure translations, pure rotations and Lorentz-boosts in the $x^1$ direction. So it is enough to prove the lemma for pure translations, pure rotations and Lorentz-boosts in the $x^1$ direction separately. This can be done by straightforward calculations. □

The electromagnetic potential in the reference frame $\tilde{K}$ is given by

$$\tilde{A}_\mu(\tilde{x}) = (\Lambda^{-1})^\nu{}_\mu A_\nu(\Lambda^{-1}(\tilde{x} - a)).$$

So we define

$$\tilde{\hat{\mathcal{H}}} = \left\{ \tilde{\Psi} : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4 \,\middle|\, \tilde{\Psi} \text{ cont. diff.}, \left( \mathrm{i}\gamma^\mu \partial_\mu - \frac{e}{c\hbar} \gamma^\mu \tilde{A}_\mu - \frac{mc}{\hbar} \right) \tilde{\Psi}(\tilde{x}) = 0, \right.$$
$$\left. \|\tilde{\Psi} \circ \sigma_\lambda\|_\lambda < \infty, \lim_{|\vec{u}| \to \infty} |\vec{u}|^3 |\tilde{\Psi} \circ \sigma_\lambda(\vec{u})|^2 = 0 \qquad \forall \lambda \in \mathcal{P} \right\}. \tag{7}$$

A scalar product $\langle\cdot|\cdot\rangle_{\tilde{\hat{\mathcal{H}}}}$ between two elements of $\tilde{\hat{\mathcal{H}}}$ and a completion $(\tilde{\mathcal{H}}, \langle\cdot|\cdot\rangle_{\tilde{\mathcal{H}}})$ can be constructed in the same way as in the previous section.

Let the quantum state in the reference frame $K$ be $\Psi \in \hat{\mathcal{H}}$. Then the quantum state in the reference frame $\tilde{K}$ is defined to be

$$\tilde{\Psi}(\tilde{x}) = S(\Lambda)\Psi(\Lambda^{-1}(\tilde{x} - a)).$$

We get the following theorem.

**Theorem 7.** *Let $\Psi \in \hat{\mathcal{H}}$, then $\tilde{\Psi}(\tilde{x}) = S(\Lambda)\Psi(\Lambda^{-1}(\tilde{x} - a)) \in \tilde{\hat{\mathcal{H}}}$.*

**Proof.** It is clear that $\tilde{\Psi}$ is continuous differentiable and that $\tilde{\Psi}$ is a solution of the Dirac equation with external field $\tilde{A}_\mu$ (see, e.g., [16]). The third condition in (7) is clear because of the lemma. The last condition can be proved by simple calculations. Again, it is enough to do this only for pure translations, pure rotations and Lorentz-boosts in the $x^1$ direction separately. □

Now, we look at the general case $F \in \mathcal{H}$. Let us define an operator $W_{(\Lambda,a)} : \mathcal{H} \to \tilde{\mathcal{H}}$:

$$(W_{(\Lambda,a)}F)(\tilde{x}) := S(\Lambda)F(\Lambda^{-1}(\tilde{x} - a)) \qquad \forall F \in \mathcal{H}.$$

This operator is well defined. Using these transformation rules, the scalar product is covariant. Its value is the same in all reference frames. Or in other words, the operator $W_{(\Lambda,a)}$ is unitary. All these will be proved by the next theorem.

**Theorem 8.**

*(i) Let $F \in \mathcal{H}$, then $\tilde{F}(\tilde{x}) = (W_{(\Lambda,a)}F)(\tilde{x}) \in \tilde{\mathcal{H}}$.*
*(ii) Let $F_1, F_2 \in \mathcal{H}$, define $\tilde{F}_1 = W_{(\Lambda,a)}F_1$, $\tilde{F}_2 = W_{(\Lambda,a)}F_2$ then*

$$\langle \tilde{F}_1 | \tilde{F}_2 \rangle_{\tilde{\mathcal{H}}} = \langle W_{(\Lambda,a)}F_1 | W_{(\Lambda,a)}F_2 \rangle_{\tilde{\mathcal{H}}} = \langle F_1 | F_2 \rangle_{\mathcal{H}}.$$

**Proof.** (i) Because of the lemma, we get

$$\|\tilde{F} \circ \sigma_\lambda\|_\lambda = \|F \circ \sigma_{\mu(\lambda)}\|_{\mu(\lambda)} < \infty$$

for all $\lambda \in \mathcal{P}$.

Only the existence of the sequence $\tilde{\Psi}_n \in \tilde{\hat{\mathcal{H}}}$ is left to be proved. Since $F \in \mathcal{H}$, there exists a sequence $\Psi_n \in \hat{\mathcal{H}}$ with $\|(F - \Psi_n) \circ \sigma_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0$ uniformly for all $\lambda \in \mathcal{P}$. Let $\tilde{\Psi}_n(\tilde{x}) = S(\Lambda)\Psi_n(\Lambda^{-1}(\tilde{x} - a))$. $\tilde{\Psi}_n \in \tilde{\hat{\mathcal{H}}}$ because of theorem 7. Since $\|(F - \Psi_n) \circ \sigma_\lambda\|_\lambda \overset{n\to\infty}{\longrightarrow} 0$ uniformly for all $\lambda \in \mathcal{P}$, we also get

$$\|(\tilde{F} - \tilde{\Psi}_n) \circ \sigma_\lambda\|_\lambda \overset{\text{see lemma}}{=} \|(F - \Psi_n) \circ \sigma_{\mu(\lambda)}\|_{\mu(\lambda)} \overset{n\to\infty}{\longrightarrow} 0$$

uniformly for all $\lambda \in \mathcal{P}$. We have indeed $\tilde{F} \in \tilde{\mathcal{H}}$.
(ii)

$$\begin{aligned}\langle \tilde{F}_1 | \tilde{F}_2 \rangle_{\tilde{\mathcal{H}}} &= \langle \tilde{F}_1 \circ \sigma_\lambda | \tilde{F}_2 \circ \sigma_\lambda \rangle_\lambda \qquad \text{arbitrary} \quad \lambda \in \mathcal{P} \\ &\overset{\text{see lemma}}{=} \langle F_1 \circ \sigma_{\mu(\lambda)} | F_2 \circ \sigma_{\mu(\lambda)} \rangle_{\mu(\lambda)} \\ &= \langle F_1 | F_2 \rangle_{\mathcal{H}}.\end{aligned}$$

□

Now, we examine the situation if we charge conjugate the system $K \to K^C$. We define

$$\hat{\mathcal{H}}^C = \left\{ \Psi^C : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}^4 \,\middle|\, \Psi^C \text{ cont. diff.}, \left(\mathrm{i}\gamma^\mu \partial_\mu + \frac{e}{c\hbar}\gamma^\mu A_\mu - \frac{mc}{\hbar}\right)\Psi^C(x) = 0, \right.$$
$$\left. \|\Psi^C \circ \sigma_\lambda\|_\lambda < \infty, \lim_{|\vec{u}|\to\infty} |\vec{u}|^3 |\Psi^C \circ \sigma_\lambda(\vec{u})|^2 = 0 \qquad \forall \lambda \in \mathcal{P} \right\}. \tag{8}$$

Again a scalar product $\langle \cdot | \cdot \rangle_{\hat{\mathcal{H}}^C}$ between two elements of $\hat{\mathcal{H}}^C$ and a completion $(\mathcal{H}^C, \langle \cdot | \cdot \rangle_{\mathcal{H}^C})$ can be constructed in the same way as in the previous section.

It is well known that in any representation of the $\gamma$-matrices there must exist a matrix $C$ which satisfies

$$C\gamma^{\mu T}C^{-1} = -\gamma^\mu \tag{9}$$

(see, e.g., [16]). In addition, we want to use only representations of the $\gamma$-matrices for which there exist an unitary matrix $C$ satisfying (9). (This is true, e.g., in the Dirac representation with $C = \mathrm{i}\gamma^2\gamma^0$.)

The following theorem expresses the relation between the spaces $\mathcal{H}$ and $\mathcal{H}^C$. It can be proved by straightforward calculations.

**Theorem 9.**

*(i) Let* $\Psi \in \hat{\mathcal{H}}$, *then* $\Psi^C = C\gamma^{0T}\Psi^* \in \hat{\mathcal{H}}^C$.

*(ii) Let* $F \in \mathcal{H}$, *then* $F^C = C\gamma^{0T}F^* \in \mathcal{H}^C$.

*(iii) Let* $F_A, F_B \in \mathcal{H}$, *let* $F_A^C = C\gamma^{0T}F_A^*$, $F_B^C = C\gamma^{0T}F_B^*$, *then* $\left\langle F_A^C \middle| F_B^C \right\rangle_{\mathcal{H}^C} = \langle F_B | F_A \rangle_{\mathcal{H}}$.

## 4. Events generating algorithm (ideal, infinitesimal short measurements)

In the previous sections, we have precisely defined the state of the total system and examined some of its properties. We are now in a position to present the proper-time evolution of the system state. More precisely, we will postulate algorithms which generate events, i.e. irreversible changes of the classical state. Because we know that the set of quantum states is indeed a Hilbert space, we can use the well-known formulation in the Hilbert space framework. In this section, we formulate an algorithm to describe ideal measurements of infinitesimal short duration. In principle, we rewrite the standard reduction postulate of the non-relativistic quantum mechanics by replacing $t$ with $\tau$ and using our Hilbert space of 'solutions'. But the quantum state should change only in the case of measurement, i.e. the 'Hamiltonian' is zero. Performing a measurement a reduction should happen. The reduction changes the wavefunction on the whole spacetime (not, e.g., only along a hyperplane) in such a way that the resulting wavefunction is again a (generalized) solution of the Dirac equation on the spacetime continuum. In this way, we get a covariant algorithm playing the role of the standard reduction postulate in non-relativistic quantum mechanics.

We name the reference frame $K$. Let the particle be prepared at proper time $\tau_0$ at a spacetime point $z_0$.

There should be $n$ measurements, which happen at proper times $\tau_i$ at spacetime points $z_i$, $i = 1, \ldots, n$. The $i$th measurement is represented by an observable $M_i$ with

$$M_i = \sum_j \lambda_{i,j} |\Phi_{i,j}\rangle\langle\Phi_{i,j}|$$

$\Phi_{i,j} \in \mathcal{H}$, $1 = \sum_j |\Phi_{i,j}\rangle\langle\Phi_{i,j}|$, $\langle\Phi_{i,j}|\Phi_{i,k}\rangle_{\mathcal{H}} = \delta_{j,k}$ and $\lambda_{i,j} \in \mathbb{R}$.

We assume that $\tau_0 < \tau_1 < \cdots < \tau_n$. We want to preserve a weak kind of order, so we demand the following: no event (e.g., preparation, measurement or detection) can take place in the backward light-cone of the previous event:

$$\left(\|z_{j+1} - z_j\|^2 \geqslant 0 \text{ and } z_j^0 < z_{j+1}^0\right) \qquad \text{or} \qquad \left(\|z_{j+1} - z_j\|^2 < 0\right) \qquad \forall j = 0, 1, \ldots, n-1$$

with $\|x\|^2 = \|(x^0, \vec{x})\|^2 = (x^0)^2 - |\vec{x}|^2$ being the Minkowski distance. Or in other words, let two successive events happen at spacetime points $z_j = \left(z_j^0, \vec{z}_j\right)$ and $z_{j+1} = \left(z_{j+1}^0, \vec{z}_{j+1}\right)$, then there must exist a Poincaré transformation $(\Lambda, a)$ $\left(\Lambda \in L_+^{\uparrow}\right)$ such that

$$\Lambda^0{}_\mu z_j^\mu + a^0 = \tilde{z}_j^0 \overset{!}{<} \tilde{z}_{j+1}^0 = \Lambda^0{}_\mu z_{j+1}^\mu + a^0.$$

There must exist a reference frame in which the time of the first event $\tilde{z}_j^0$ is earlier than the time of the second event $\tilde{z}_{j+1}^0$.

Now we start with the formulation of a relativistic reduction postulate for ideal measurements. Let $(\omega_\tau, \Psi_\tau)$ be the state of the total system.

(i) The particle is prepared at proper time $\tau_0$ at spacetime point $z_0$. The quantum state is given by $\Psi_{\tau_0}$ with $\|\Psi_{\tau_0}\|^2_{\mathcal{H}} = 1$ and the classical state is $\omega_{\tau_0} = 0$. Let $i = 1$.

(ii) The quantum and the classical states change only in the case of measurement. They have no $\tau$-dependence if there is no measurement:

$$(\omega_\tau, \Psi_\tau) = \left(\omega_{\tau_{i-1}}, \Psi_{\tau_{i-1}}\right)$$

for $\tau_{i-1} \leqslant \tau \leqslant \tau_i$.

(iii) The $i$th measurement takes place at proper time $\tau_i$ at spacetime point $z_i$. We choose the measurement result $\lambda_{i,j}$ with probability

$$p(\lambda_{i,j}) = \left|\left\langle \Phi_{i,j} \middle| \Psi_{\tau_i} \right\rangle_{\mathcal{H}}\right|^2.$$

If $\lambda_{i,j}$ is the received measurement result, the state of the total system changes in the following way:

$$\left(\omega_{\tau_i}, \Psi_{\tau_i}\right) \longrightarrow (j, \Phi_{i,j}).$$

(iv) Let $i \to i + 1$ and go to step (ii).

We want to examine what this algorithm looks like in another reference frame. Let $\tilde{K}$ be a reference frame which is connected to $K$ by a Poincaré transformation $(\Lambda, a)$ with $\Lambda \in L^{\uparrow}_{+}$.

In $\tilde{K}$, the situation is described in this way: the particle is prepared at $\tau_0$ in $\tilde{z}_0 = \Lambda z_0 + a$ with initial quantum state

$$\tilde{\Psi}_{\tau_0}(\tilde{x}) = \left(W_{(\Lambda,a)}\Psi_{\tau_0}\right)(\tilde{x}) = S(\Lambda)\Psi_{\tau_0}(\Lambda^{-1}(\tilde{x} - a))$$

with $\|\tilde{\Psi}_{\tau_0}\|^2_{\tilde{\mathcal{H}}} = 1$ (the operator $W_{(\Lambda,a)}$ is unitary). The measurement $i$ at proper time $\tau_i$ happens at $\tilde{z}_i = \Lambda z_i + a$ and is represented by

$$\tilde{M}_i = W_{(\Lambda,a)} M_i W^{+}_{(\Lambda,a)} = \sum_j \lambda_{i,j} |\tilde{\Phi}_{i,j}\rangle\langle\tilde{\Phi}_{i,j}|$$

with $\tilde{\Phi}_{i,j} = W_{(\Lambda,a)}\Phi_{i,j}$. It is true that $1 = \sum_j |\tilde{\Phi}_{i,j}\rangle\langle\tilde{\Phi}_{i,j}|$ and $\langle\tilde{\Phi}_{i,j}|\tilde{\Phi}_{i,k}\rangle_{\tilde{\mathcal{H}}} = \delta_{j,k}$.

If we apply the algorithm in $\tilde{K}$ and if we choose the same random numbers, then we get the same measurement results as those we get if we apply the algorithm in $K$, because

$$\tilde{p}(\lambda_{i,j}) = \left|\left\langle W_{(\Lambda,a)}\Phi_{i,j} \middle| W_{(\Lambda,a)}\ \Psi_{\tau_i} \right\rangle_{\tilde{\mathcal{H}}}\right|^2 = \left|\left\langle \Phi_{i,j} \middle|\ \Psi_{\tau_i} \right\rangle_{\mathcal{H}}\right|^2 = p(\lambda_{i,j}).$$

The system state $(\omega_\tau, \Psi_\tau)$ in the reference frame $K$ and the system state $(\tilde{\omega}_\tau, \tilde{\Psi}_\tau)$ in the reference frame $\tilde{K}$ are always connected in the following way:

$$(\tilde{\omega}_\tau, \tilde{\Psi}_\tau) = (\omega_\tau, W_{(\Lambda,a)}\Psi_\tau).$$

The above algorithm describing ideal, infinitesimal short measurements is covariant.

Now we consider the charge-conjugated system $K^C$. We set

$$\Psi^C_{\tau_0} := C\gamma^{0T}\Psi^*_{\tau_0} \qquad \Phi^C_{i,j} := C\gamma^{0T}\Phi^*_{i,j}.$$

The charge-conjugated observables are defined by

$$M^C_i = C\gamma^{0T}M^*_i\gamma^{0T}C^{+} = \sum_j \lambda_{i,j} \left|\Phi^C_{i,j}\right\rangle\left\langle\Phi^C_{i,j}\right|.$$

The complex conjugated operator $M^*_i$ is defined by $M^*_i\Psi := (M_i\Psi^*)^*$.

If we execute the algorithm in the charge-conjugated system $K^C$ or if we execute the algorithm in the normal system $K$, both will result in the same events (if we choose the same random numbers), because

$$p^C(\lambda_{i,j}) = \left|\left\langle \Phi^C_{i,j} \middle|\ \Psi^C_{\tau_i} \right\rangle_{\mathcal{H}^C}\right|^2 = |\langle\Psi_{\tau_i}|\Phi_{i,j}\rangle_{\mathcal{H}}|^2 = p(\lambda_{i,j}).$$

The system state $(\omega_\tau, \Psi_\tau)$ in $K$ and the system state $\left(\omega_\tau^C, \Psi_\tau^C\right)$ in $K^C$ are always connected by

$$\left(\omega_\tau^C, \Psi_\tau^C\right) = \left(\omega_\tau, C\gamma^{0T}\Psi_\tau^*\right).$$

We demand that the algorithm applied in the 'charge-conjugated world' or applied in the 'normal world' describes the same physical situation.

We end this section with the derivation of an important relationship between the standard reduction postulate used with the Dirac equation and the above algorithm: the standard reduction postulate formulated in a (preferred) fixed reference frame can be rewritten as a special case of the above algorithm. Especially, the standard reduction postulate used in a fixed reference frame gives the same probabilities as the above (covariant) algorithm.

We choose the (preferred) fixed reference frame. We assume that the electromagnetic potential $A_\mu$ is time-independent in this frame. Now, we define

$$H_D = -\mathrm{i}c\hbar\gamma^0\gamma^k\frac{\partial}{\partial x^k} + e\gamma^0\gamma^\mu A_\mu + \gamma^0 mc^2.$$

Let $U_t \equiv U_{((ct,\vec{0}),\vec{0},\vec{0})}$ (see (6)), so that $(U_t\Psi)(\vec{u}) = \Psi(ct,\vec{u})$. We are now in a position to prove our claim.

Let the wavefunction be $\psi_0$ at time $t = 0$ with $\|\psi_0\|_{L_2(\mathbb{R}^3)^4} = 1$. We assume measurements happening at times $t_1, \ldots, t_n$ with $0 < t_1 < t_2 < \cdots < t_n$. The measurement $i$ is represented by an observable $m_i$ with

$$m_i = \sum_j \lambda_{i,j}|\phi_{i,j}\rangle\langle\phi_{i,j}|$$

and $1 = \sum_j |\phi_{i,j}\rangle\langle\phi_{i,j}|$, $\langle\phi_{i,j}|\phi_{i,k}\rangle_{L_2(\mathbb{R}^3)^4} = \delta_{j,k}$ and $\lambda_{i,j} \in \mathbb{R}$.

Next, we describe this situation in the framework of our formalism. Let

$$\Psi_0 := U_0^{-1}\psi_0.$$

We get $\|\Psi_0\|_{\mathcal{H}} = 1$. We define $n$ measurements happening at proper times $\tau_i := t_i$ at spacetime points $z_i = (ct_i, \vec{y}_i)$. $\vec{y}_i$ can be chosen arbitrarily. The measurements are represented by observables $M_i$ with

$$M_i := U_{ct_i}^{-1} m_i U_{ct_i} = \sum_j \lambda_{i,j}|\Phi_{i,j}\rangle\langle\Phi_{i,j}|$$

with $\Phi_{i,j} = U_{ct_i}^{-1}\phi_{i,j}$. Note that $1 = \sum_j |\Phi_{i,j}\rangle\langle\Phi_{i,j}|$ and $\langle\Phi_{i,j}|\Phi_{i,k}\rangle_{\mathcal{H}} = \delta_{j,k}$.

We execute the standard reduction postulate (SR) and the above algorithm (AL).

(i) SR. At time $t = 0$ the wavefunction is $\psi_0$.
AL. At $\tau = 0$ the state of the quantum part is $\Psi_0$ with $\psi_0 = U_0\Psi_0$.

(ii) SR. Until $t = t_1$, the time evolution of the wavefunction is given by

$$\psi(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}tH_D\right)\psi_0.$$

AL. The state of the quantum part does not change until $\tau = \tau_1 = t_1$:

$$\Psi_\tau = \Psi_0.$$

The following relationship between $\psi(t)$ and $\Psi_0$ is fulfilled for $0 \leqslant t \leqslant t_1$:

$$\psi(t) = \exp\left(-\frac{\mathrm{i}}{\hbar}tH_D\right)U_0\Psi_0 = U_t\Psi_0 = U_t\Psi_t.$$

(iii) SR. At $t = t_1$, the first measurement happens. The probability for the result $\lambda_{1,j}$ is given by

$$p_{1,j} = |\langle \phi_{1,j} | \psi(t_1) \rangle_{L_2(\mathbb{R}^3)^4}|^2.$$

AL. At $\tau = \tau_1 = t_1$, the first measurement happens. The probability for the result $\lambda_{1,j}$ is given by

$$\begin{aligned} p_{1,j} &= |\langle \Phi_{1,j} | \Psi_0 \rangle_{\mathcal{H}}|^2 \\ &= \left|\left\langle U_{t_1}^{-1} \phi_{1,j} \right| U_{t_1}^{-1} \ \psi(t_1) \rangle_{\mathcal{H}}\right|^2 \\ &= |\langle \phi_{1,j} | \psi(t_1) \rangle_{L_2(\mathbb{R}^3)^4}|^2. \end{aligned}$$

(iv) SR. The result should be $\lambda_{1,j}$. Then, the following change of the wavefunction happens

$$\psi(t_1) \longrightarrow \phi_{1,j} = U_{t_1} \Phi_{1,j}.$$

AL. The result should be $\lambda_{1,j}$. Then, the following change of the wavefunction happens

$$\Psi_0 \longrightarrow \Phi_{1,j} = U_{t_1}^{-1} \phi_{1,j}.$$

The algorithm continues with the other measurements.

We want to underline two facts. First the probabilities resulting from the standard reduction postulate and the probabilities resulting from the above algorithm are equal. Additionally, it is true for all $t \geqslant 0$ that

$$\psi(t) = U_t \Psi_t.$$

In the same way, we can derive the relationship between our algorithm and the algorithm for ideal, localized measurements proposed in the approach of Breuer and Petruccione [9]. Let a foliation of spacetime be a family of space-like hyperplanes $\sigma_{\lambda(\tau)}$ which represent the equal-time hyperplanes of an observer moving along a time-like trajectory. Their algorithm with a fixed foliation $\sigma_{\lambda(\tau)}$ can be formulated as a special case of our algorithm. On the other hand, if a foliation $\sigma_{\lambda(\tau)}$ exists with $z_j \in \sigma_{\lambda(\tau_j)}\ \forall j$ then our algorithm can be formulated as a special case of their algorithm with the foliation $\sigma_{\lambda(\tau)}$. Both algorithms will give the same measurement results if we choose the same random numbers. However, the physical ideas are different. In the formalism of Breuer and Petruccione the wavefunction cannot in general be written as a function on the spacetime continuum but in our formalism the wavefunction is always a function on the spacetime continuum.

## 5. Events generating algorithm (detections of the particle)

In this section, we formulate an algorithm for modelling continuous relativistic measurements, indeed we will propose in the following an algorithm to simulate detections of the particle. In principle, we do this by rewriting the algorithm of EEQT by replacing $t$ with $\tau$ and using our Hilbert space of 'solutions'. Again, the quantum state should change only in the case of measurement, i.e. the 'Hamiltonian' is zero.

We label the reference frame $K$. The particle is prepared at proper time $\tau_0$ in a point $x_0 = \left(x_0^0, \vec{x_0}\right)$.

We consider $n$ detectors with trajectories $z_j(\tau)$, $j = 1, \ldots, n$. The trajectories start at proper time $\tau = \tau_0$ from the backward light-cone of the spacetime point of the 'preparation event':

$$\|x_0 - z_j(\tau_0)\|^2 = 0 \qquad z_j^0(\tau_0) \leqslant x_0^0.$$

We allow detections which happen in the past of the preparation time. But we do not allow detections if the detection spacetime point is located in the backward light-cone of the spacetime point of the preparation event.

Each detector is characterized by operators $G_j(\tau)$ mapping (generalized) solutions on (generalized) solutions. Let $G_j^+(\tau)$ be the adjoint operators. The total coupling between the quantum and the classical systems is given by $\Lambda(\tau) := \sum_j G_j^+(\tau)G_j(\tau)$.

Let $(\omega_\tau, \Psi_\tau)$ be the state of the total system. We define the following algorithm:

(i) The particle is prepared in a spacetime point $x_0$ at proper time $\tau = \tau_0$. The quantum state is $\Psi_{\tau_0}$ with $\|\Psi_{\tau_0}\|_{\mathcal{H}}^2 = 1$ and the classical state is $\omega_{\tau_0} = 0$.

(ii) Choose a uniform random number $r \in [0, 1]$.

(iii) Propagate the quantum state forward in proper time by solving

$$\frac{\partial}{\partial\tau}\Psi_\tau = -\frac{1}{2}\Lambda(\tau)\Psi_\tau \tag{10}$$

until $\tau = \tau_1$, where $\tau_1$ is defined by

$$1 - \|\Psi_{\tau_1}\|_{\mathcal{H}}^2 = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \langle\Psi_\tau|\Lambda\Psi_\tau\rangle_{\mathcal{H}} = r.$$

Let $\omega_\tau = \omega_{\tau_0}$ until $\tau = \tau_1$, a detection happens at proper time $\tau = \tau_1$.

(iv) We choose the detector $k$, which detects the particle, with probability

$$p_k = \frac{1}{N}\|G_k(\tau_1)\Psi_{\tau_1}\|_{\mathcal{H}}^2$$

with $N = \sum_j \|G_j(\tau_1)\Psi_{\tau_1}\|_{\mathcal{H}}^2$.

(v) Let $l$ be the detector which detects the particle. The detection happens at the point $z_l(\tau_1)$. The detection induces the following change of the states:

$$\left(\omega_{\tau_1}, \Psi_{\tau_1}\right) \longrightarrow \left(l, \frac{G_l(\tau_1)\Psi_{\tau_1}}{\|G_l(\tau_1)\Psi_{\tau_1}\|_{\mathcal{H}}}\right).$$

The algorithm can start again perhaps with other detectors at position (ii).

We want to examine what this algorithm looks like in another reference frame. Let $\tilde{K}$ be the reference frame which is connected to $K$ by a Poincaré transformation $(\Lambda, a)$ with $\Lambda \in L_+^\uparrow$.

In $\tilde{K}$, the situation can be described as follows: the particle is prepared at $\tau_0$ in $\tilde{x}_0 = \Lambda x_0 + a$ with initial quantum state

$$\tilde{\Psi}_{\tau_0}(\tilde{x}) = \left(W_{(\Lambda,a)}\Psi_{\tau_0}\right)(\tilde{x}) = S(\Lambda)\Psi_{\tau_0}(\Lambda^{-1}(\tilde{x} - a))$$

with $\|\tilde{\Psi}_{\tau_0}\|_{\tilde{\mathcal{H}}}^2 = 1$ (because the operator $W_{(\Lambda,a)}$ is unitary). The trajectories of the detectors are $\tilde{z}_i = \Lambda z_i + a$, and the detectors are characterized by

$$\tilde{G}_j(\tau) = W_{(\Lambda,a)}G_j(\tau)W_{(\Lambda,a)}^+.$$

We get $\tilde{\Lambda}(\tau) = \sum_j \tilde{G}_j^+(\tau)\tilde{G}_j(\tau) = W_{(\Lambda,a)}\Lambda(\tau)W_{(\Lambda,a)}^+$.

Note, that if $\Psi_\tau$ is a solution of (10) then $\tilde{\Psi}_\tau := W_{(\Lambda,a)}\Psi_\tau$ is a solution of

$$\frac{\partial}{\partial\tau}\tilde{\Psi}_\tau = W_{(\Lambda,a)}\frac{\partial}{\partial\tau}\Psi_\tau = -\frac{1}{2}W_{(\Lambda,a)}\Lambda(\tau)\Psi_\tau = -\frac{1}{2}W_{(\Lambda,a)}\Lambda(\tau)W_{(\Lambda,a)}^+\tilde{\Psi}_\tau = -\frac{1}{2}\tilde{\Lambda}(\tau)\tilde{\Psi}_\tau.$$

This result implies that the algorithm executed in the reference frame $\tilde{K}$ will give the same detections as the algorithm executed in $K$ (if we choose the same random numbers). The spacetime points of the detections in the two reference frames are connected by the Poincaré transformation $(\Lambda, a)$.

The system state $(\omega_\tau, \Psi_\tau)$ in the reference frame $K$ and the system state $(\tilde{\omega}_\tau, \tilde{\Psi}_\tau)$ in the reference frame $\tilde{K}$ are always connected in the following way:

$$(\tilde{\omega}_\tau, \tilde{\Psi}_\tau) = (\omega_\tau, W_{(\Lambda,a)}\Psi_\tau).$$

The algorithm modelling detections of the particle is indeed covariant.

Now we consider the charge-conjugated system $K^C$. Let

$$\Psi^C_{\tau_0} := C\gamma^{0^T}\Psi^*_{\tau_0} \in \mathcal{H}^C.$$

The charge-conjugated coupling is given by

$$G^C_j(\tau) = C\gamma^{0^T}G^*_j(\tau)\gamma^{0^T}C^+ \tag{11}$$

with $G^*_j(\tau)\Psi := (G_j(\tau)\Psi^*)^*$. Let

$$\Lambda^C(\tau) = \sum_j G^{C+}_j(\tau)G^C_j(\tau) = C\gamma^{0^T}\Lambda^*(\tau)\gamma^{0^T}C^+.$$

Note, that if $\Psi_\tau$ is a solution of (10) then $\Psi^C_\tau \equiv C\gamma^{0^T}\Psi^*_\tau$ is a solution of

$$\frac{\partial}{\partial\tau}\Psi^C_\tau = C\gamma^{0^T}\frac{\partial}{\partial\tau}\Psi^*_\tau = -\frac{1}{2}C\gamma^{0^T}\Lambda^*(\tau)\gamma^{0^T}C^+\Psi^*_\tau = -\frac{1}{2}\Lambda^C(\tau)\Psi^C_\tau.$$

We also note that

$$G^C_j(\tau)\Psi^C_\tau = C\gamma^{0^T}G^*_j(\tau)\gamma^{0^T}C^+C\gamma^{0^T}\Psi^*_\tau = C\gamma^{0^T}(G_j(\tau)\Psi_\tau)^* \equiv (G_j(\tau)\Psi_\tau)^C.$$

A corollary of this fact is $\left\|G^C_j(\tau)\Psi^C_\tau\right\|^2_{\mathcal{H}^C} = \|G_j(\tau)\Psi_\tau\|^2_{\mathcal{H}}$ and $\left\langle\Psi^C_\tau\middle|\Lambda^C(\tau)\Psi^C_\tau\right\rangle_{\mathcal{H}^C} = \langle\Psi_\tau|\Lambda(\tau)\Psi_\tau\rangle_{\mathcal{H}}$.

We can conclude: if we start with $\Psi_{\tau_0} \in \mathcal{H}$ and operators $G_j(\tau)$, then the algorithm will give the same results as if we start with $\Psi^C_{\tau_0} = C\gamma^{0^T}\Psi^*_{\tau_0} \in \mathcal{H}^C$ and operators $G^C_j(\tau)$ defined in (11) (if we choose the same random numbers).

The state $(\omega_\tau, \Psi_\tau)$ in the normal system $K$ and the state $\left(\omega^C_\tau, \Psi^C_\tau\right)$ in the charge-conjugated system $K^C$ are again connected by

$$\left(\omega^C_\tau, \Psi^C_\tau\right) = \left(\omega_\tau, C\gamma^{0^T}\Psi^*_\tau\right).$$

Again, we demand that the algorithm applied in the 'charge-conjugated world' or applied in the 'normal world' describes the same physical situation.

In the last part of this section, we examine the non-relativistic limit of the above algorithm and prove heuristically that the non-relativistic limit reduces to the algorithm of the non-relativistic EEQT. To establish this fact, we define

$$\Omega(\tau, \vec{x}) := (U_\tau\Psi_\tau)(\vec{x}) = \Psi_\tau(c\tau, \vec{x})$$

with $\Psi_\tau$ being a solution of (10) (we recall that $U_t \equiv U_{((ct,\vec{0}),\vec{0},\vec{0})}$) and we assume that $\Psi_\tau \in \hat{\mathcal{H}}\ \forall\tau$. We get

$$\begin{aligned}
\mathrm{i}\hbar\frac{\partial}{\partial\tau}\Omega(\tau,\vec{x}) &= \mathrm{i}\hbar c\left(\frac{\partial}{\partial x^0}\Psi_\tau\right)(c\tau,\vec{x}) + \mathrm{i}\hbar\frac{\partial\Psi_\tau}{\partial\tau}(c\tau,\vec{x}) \\
&= H_D\Psi_\tau(c\tau,\vec{x}) - \mathrm{i}\frac{\hbar}{2}(U_\tau\Lambda(\tau)\Psi_\tau)(\vec{x}) \\
&= H_D\Omega(\tau,\vec{x}) - \mathrm{i}\frac{\hbar}{2}\left(\sum_j \underbrace{\left[U_\tau G^+_j(\tau)U^{-1}_\tau\right]}_{=:g^+_j(\tau)}\underbrace{\left[U_\tau G_j(\tau)U^{-1}_\tau\right]}_{=:g_j(\tau)}\right)\Omega(\tau,\vec{x}).
\end{aligned} \tag{12}$$

We examine the non-relativistic limit of (12) using the assumption (in analogy with calculations of the non-relativistic limit of the Dirac equation, see e.g., [16])

$$\Omega(\tau,\vec{x}) = \exp\left(-\mathrm{i}\frac{mc^2}{\hbar}\tau\right)\begin{pmatrix}\phi\\ \chi\end{pmatrix}. \tag{13}$$

Furthermore, we assume that

$$g_j(\tau) = \begin{pmatrix} g_{j,1}(\tau) & 0\\ 0 & g_{j,2}(\tau)\end{pmatrix}.$$

Inserting (13) in (12), we take the limit $c\to\infty$ but we keep $\frac{e}{c}A^k$. In this way, we obtain the modified equation of the non-relativistic EEQT (see, for example, [3])

$$\mathrm{i}\hbar\frac{\partial}{\partial\tau}\phi = \left[\frac{1}{2m}\sum_l\left(\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^l} - \frac{e}{c}A^l\right)^2 - \frac{e\hbar}{2mc}\vec{\hat{\sigma}}\vec{B} + eA^0 - \mathrm{i}\frac{\hbar}{2}\sum_j g^+_{j,1}(\tau)g_{j,1}(\tau)\right]\phi$$

with $\hat{\sigma}^k$ being the Pauli matrices. We note that

$$\langle\Psi_\tau|\Psi_\tau\rangle_{\mathcal{H}} = \int \mathrm{d}\vec{x}\ \Omega^+(\tau,\vec{x})\Omega(\tau,\vec{x}) \overset{c\to\infty}{\longrightarrow} \int \mathrm{d}\vec{x}\,\phi^+(\tau,\vec{x})\phi(\tau,\vec{x}).$$

If we set $t := \tau$, we see immediately that the algorithm of the EEQT is the non-relativistic limit of the above relativistic algorithm.

Breuer and Petruccione also proposed an algorithm for continuous measurements in their approach [10]. Under some conditions (because, e.g., space-like separated operators must commute in their formalism but we do not require this explicitly), their algorithm with a fixed foliation can again be formulated as a special case of our algorithm and vice versa. But again the physical ideas are different.

## 6. Summary

In this paper, we have presented an alternative version of a relativistic extension of the event-enhanced quantum theory (EEQT). It describes one massive spin-$\frac{1}{2}$ particle.

We use the idea of an additional time, the proper time, which is invariant in all reference frames (in analogy with the relativistic extension of Blanchard and Jadczyk [14]).

The total system consists of a quantum part and a classical part analogous to EEQT. A pure state $\omega_\tau$ of the classical part at a proper time $\tau$ is a number ($\omega_\tau\in\mathbb{N}_0$). A pure state $\Psi_\tau$ of the quantum part at a proper time $\tau$ is (heuristically) a solution of the Dirac equation. We have proved that the solutions of the Dirac equation can be extended to a separable Hilbert space with a positive-definite scalar product. The pure quantum states are the elements of this Hilbert space and still functions on the spacetime continuum. An important property of a quantum state $\Psi_\tau$ is that it is uniquely given by its projection onto a space-like hyperplane.

The advantage of a positive-definite scalar product must be paid for with a more complicated Hilbert space compared to the relativistic extension of Blanchard and Jadczyk [14]. In that extension, the Hilbert space is simpler but they use an indefinite scalar product.

The transformation rules of a system state (if we change the reference frame) have been presented. They are chosen in such a way that the scalar product between two quantum states is independent of the reference frame.

First, we have postulated a covariant algorithm to simulate ideal, infinitesimal short measurements. We have shown that the (non-covariant) standard reduction postulate formulated in a (preferred) fixed reference frame can be rewritten as a special case of our (covariant) algorithm.

Second, we have postulated a covariant algorithm to simulate detections of a particle. We have shown that the non-relativistic limit of this algorithm reduces to the PDP algorithm of the non-relativistic EEQT.

Moreover, we have shown that both algorithms are invariant by charge conjugation.

Both algorithms are also correlated with those proposed by Breuer and Petruccione [7–11]. The algorithms can be written (under some conditions) as special cases of those proposed by Breuer and Petruccione and vice versa. But the physical ideas behind them are different. While in our formalism the wavefunction is always a function on the spacetime continuum, this is in general not true in the approach of Breuer and Petruccione.

We want to end this paper with a summary of the properties of an event in our theory: an event is a change of the (pure) state of the classical part which happened at a proper time. An event can be observed without disturbing it. We demand that each event be associated with a point in spacetime. In general, if an event happens, the quantum state changes simultaneously and instantaneously over the whole spacetime. We do not want to include the principle of relativistic causality explicitly in our formalism: we even allow that an event can happen in the past of the previous (concerning proper time) event. But we want to preserve a weak kind of order, so we demand the following: no event (e.g., preparation, measurement or detection) can be created at a spacetime point which lies in the backward light-cone of that spacetime point which is associated with the previously (concerning proper time) created event. All these demands are fulfilled by the events generated by our algorithms.

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